

# PARTIAL ISOMETRIES AND THE CONJECTURE OF C. K. FONG AND S. K. TSUI

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**ABSTRACT.** We investigate some bounded linear operators  $T$  on a Hilbert space which satisfy the condition  $|T| \leq |\operatorname{Re} T|$ . We describe the maximum invariant subspace for a contraction  $T$  on which  $T$  is a partial isometry to obtain that, in certain cases, the above condition ensures that  $T$  is self-adjoint. In other words we show that the Fong-Tsui conjecture holds for partial isometries, contractive quasi-isometries, or 2-quasi-isometries, and Brownian isometries of positive covariance, or even for a more general class of operators.

## 1. INTRODUCTION AND TERMINOLOGY

For two complex Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$  we denote by  $\mathcal{B}(\mathcal{H}, \mathcal{K})$  the Banach space of all bounded linear operators from  $\mathcal{H}$  into  $\mathcal{K}$ , and  $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$  considered as a Banach algebra with  $I = I_{\mathcal{H}}$  the identity operator on  $\mathcal{H}$ . For  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ ,  $\mathcal{R}(T)$  and  $\mathcal{N}(T)$  stand for the range and the null-space of  $T$ , respectively. For a subspace  $\mathcal{G}$  of  $\mathcal{H}$  its closure is denoted by  $\overline{\mathcal{G}}$ . As usually, a closed subspace  $\mathcal{G}$  of  $\mathcal{H}$  is invariant (reducing) for  $T$  if  $T\mathcal{G} \subset \mathcal{G}$  (and  $T^*\mathcal{G} \subset \mathcal{G}$ ). Also,  $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  stands for the adjoint operator of  $T$ , while the orthogonal projection associated to a closed subspace  $\mathcal{G}$  of  $\mathcal{H}$  is denoted by  $P_{\mathcal{G}}$ , that is  $P_{\mathcal{G}} \in \mathcal{B}(\mathcal{H})$  with  $P_{\mathcal{G}}^2 = P_{\mathcal{G}} = P_{\mathcal{G}}^*$ .

An operator  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  is a *contraction* if  $\|T\| \leq 1$ , and  $T$  is a *partial isometry* when  $T^*T$  is an orthogonal projection. In particular  $T$  is an *isometry* if  $T^*T = I_{\mathcal{H}}$ , and *unitary* if  $T$  is a surjective isometry. A unitary operator  $U \in \mathcal{B}(\mathcal{H})$  with  $U^* = U$  is called a *symmetry*. A contraction  $T$  is called *pure* if  $\|Tx\| < \|x\|$  for any  $x \in \mathcal{H}$ ,  $x \neq 0$ .

Also, we say that an operator  $T \in \mathcal{B}(\mathcal{H})$  is a *m-quasi-isometry* for some integer  $m \geq 1$ , if  $T|_{\mathcal{R}(T^m)}$  is an isometry. The 1-quasi-isometries are shortly called *quasi-isometries*, such operators being firstly studied [18, 19] and latterly in [20, 21], and other articles. The generalization to *m-quasi-isometries* for  $m \geq 2$  appear in [8, 15, 16]. It was proved in [16] that a quasi-isometric contraction  $T$  is subnormal, that is it has a normal extension, hence  $T$  is hyponormal that is  $TT^* \leq T^*T$ .

As usually, for  $T \in \mathcal{B}(\mathcal{H})$  we denote the module of  $T$  by  $|T| = (T^*T)^{1/2}$ , and the real part of  $T$  by  $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ .

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An interesting conjecture formulated by C. K. Fong and S. K. Tsui in [11] says that if  $T$  satisfies the condition

$$(1.1) \quad |T| \leq |\operatorname{Re} T|$$

then  $T$  is self-adjoint.

This conjecture was partially proved in [11] in finite dimensional Hilbert spaces, in finite von Neumann algebras, and for compact operators in any Hilbert space.

Originally, C. K. Fong and V. Istrăţescu proved in [10] that  $T$  is self-adjoint if and only if  $|T|^2 \leq (\operatorname{Re} T)^2$ . In particular, by Lemma 1.5 [10] it follows that the Fong-Tsui conjecture holds for hyponormal operators.

Recently, M. H. Mortad shows in [17] that this conjecture is also true when  $T$  commutes with the partial isometry  $U$  which appears in the polar decomposition of  $\operatorname{Re} T$  ( $U$  being a symmetry on  $\overline{\mathcal{R}(\operatorname{Re} T)}$ ).

A difficulty for the solution of this conjecture is the fact that  $|T|$  and  $|\operatorname{Re} T|$  cannot be easily expressed in terms of  $T$  (and  $T^*$ ), by contrast to  $|T|^2$  and  $(\operatorname{Re} T)^2$ . An idea is to transfer the conjecture on some reducing or just invariant parts of  $T$  which may be easily expressed in  $T, T^*$ , and to investigate the condition (1.1) on each such part.

The purpose of this paper is to show that the Fong-Tsui conjecture is also true for some operators which are related to partial isometries, as well as those before.

In Section 2 we describe the maximum invariant subspace for a contraction  $T$  on which  $T$  is a partial isometry (Theorem 2.1). We use some block matrix forms for  $T$ , and we refer to the case when this subspace is just  $\mathcal{N}(T^*T - (T^*T)^2)$ , and also to a special case when this latter subspace is reducing for  $T$ .

In Section 3 we give the main result (Theorem 3.1) which asserts that for a contraction  $T$  which satisfies the condition (1.1) the fixed points of  $T^*T, TT^*$  and  $|\operatorname{Re} T|$  coincide, and their subspace reduces  $T$  to a symmetry. So, in this case the invariant partial isometric part of  $T$  in  $\mathcal{H}$  is  $\mathcal{N}(I - T^*T) \oplus \mathcal{N}(T)$ , which does not reduce  $T$ , in general.

As consequences, we derive that the Fong-Tsui conjecture holds for partial isometries, quasi-isometric or 2-quasi-isometric contractions. In the case of an  $m$ -quasi-isometric contraction  $T$  with  $m \geq 3$  we obtain that  $T$  is a symmetry on  $\overline{\mathcal{R}(T^m)}$ , this subspace being even the unitary part of  $T$  in  $\mathcal{H}$ . In this case  $T = S \oplus Q$  with  $S$  a symmetry on  $\overline{\mathcal{R}(T^m)}$  and  $Q^m = 0$ . So  $T = T^*$  if and only if  $Q = 0$ .

Another special class of non-contractive operators for which the Fong-Tsui conjecture can be shown true is given by the Brownian isometries. This class containing the Brownian unitaries was extensively studied by J. Agler and M. Stankus in [1-3]. Such operators arise naturally in the context of 2-isometries, that is of operators  $T$  on  $\mathcal{H}$  satisfying the identity  $T^{*2}T^2 - 2T^*T + I = 0$ . According to [2, Proposition 5.37], a 2-isometry  $T$  on  $\mathcal{H}$  is a Brownian isometry of covariance  $\sigma > 0$  if  $\sigma^2 = \|T^*T - I\|$  and, with respect to a decomposition

$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ ,  $T$  has a block matrix form

$$(1.2) \quad T = \begin{pmatrix} V & \sigma E \\ 0 & U \end{pmatrix},$$

where  $V$  is an isometry on  $\mathcal{H}_0$ ,  $E \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$  is an injective contraction with  $\mathcal{R}(E) \subset \mathcal{N}(V^*)$ , while  $U$  is unitary on  $\mathcal{H}_1$  such that  $UE^*E = E^*EU$ .

Finally, we remark that, under the condition (1.1) for a contraction  $T$ ,  $T^*$  has a similar block matrix form like that of  $T$  (given by Theorem 3.1), without imposing the condition (1.1) for  $T^*$ . But these matrix representations of  $T$  and  $T^*$  cannot lead to a symmetric condition (in  $T$  and  $T^*$ ) as  $T = T^*$ , from a non-symmetric one as (1.1). The Fong-Tsui conjecture remains an interesting open problem, in particular for pure contractions.

## 2. THE INVARIANT PARTIAL ISOMETRIC PART OF A CONTRACTION

The main result of this section is the following

**Theorem 2.1.** *For every contraction  $T$  on  $\mathcal{H}$  there exists the maximum subspace  $\mathcal{M}$  which is invariant for  $T$ , on which  $T$  is a partial isometry. More precisely, one has  $\mathcal{N}(T) \subset \mathcal{M} \subset \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T)$ , and  $T$  has the block matrix form*

$$(2.1) \quad T = \begin{pmatrix} W & R \\ 0 & Q \end{pmatrix}$$

on  $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ , where  $W$  is a partial isometry on  $\mathcal{M}$ ,  $R \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$  is a contraction with  $W^*R = 0$ , and  $Q$  is a contraction on  $\mathcal{M}^\perp$  such that  $\mathcal{N}(I - Q^*Q) \subset \mathcal{N}(I - T^*T)$ .

Moreover, we have

$$(2.2) \quad \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T) = \mathcal{M} \oplus \mathcal{N}(I - Q^*Q)$$

if and only if  $\mathcal{N}(I - T^*T) \cap \mathcal{M}^\perp \subset \mathcal{N}(R)$ .

*Proof.* The required subspace  $\mathcal{M}$  need to satisfy the inclusions

$$\mathcal{N}(T) \subset \mathcal{M} \subset \mathcal{N}(T^*T - (T^*T)^2) = \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T).$$

But with respect to the decomposition

$$\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T) \oplus \mathcal{H}'$$

where  $\mathcal{H}' = \overline{\mathcal{R}(T^*T - (T^*T)^2)}$ ,  $T$  has a block matrix form

$$(2.3) \quad T = \begin{pmatrix} 0 & T_0 & T'_0 \\ 0 & T_1 & T'_1 \\ 0 & T_2 & T'_2 \end{pmatrix},$$

with some appropriate contractions  $T_j$  and  $T'_j$ . In particular one has  $T_2 = P_{\mathcal{H}'}T|_{\mathcal{N}(I - T^*T)}$ , while the subspace  $\mathcal{H}_1 := \mathcal{N}(T_2)$  is invariant for  $T_2$ . Therefore  $\mathcal{N}(T) \oplus \mathcal{H}_1$  is invariant for  $T$ ,

and from the above representation of  $T$  we can infer another block matrix form for  $T$  on the decomposition  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ , where  $\mathcal{H}_0 = \mathcal{N}(T)$  and  $\mathcal{H}_2 := [\mathcal{N}(I - T^*T) \ominus \mathcal{H}_1] \oplus \mathcal{H}'$ , as follows

$$(2.4) \quad T = \begin{pmatrix} 0 & W_0 & R_0 \\ 0 & W_1 & R_1 \\ 0 & 0 & Q \end{pmatrix}.$$

Here all operators are contractions between the corresponding subspaces of the decomposition of  $\mathcal{H}$  and, in particular, for the operator  $W \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$  with the block matrix form

$$W = \begin{pmatrix} 0 & W_0 \\ 0 & W_1 \end{pmatrix},$$

we have  $W_0 = T_0|_{\mathcal{H}_1} = P_{\mathcal{H}_0}T|_{\mathcal{H}_1}$ ,  $W_1 = T_1|_{\mathcal{H}_1} = P_{\mathcal{H}_1}T|_{\mathcal{H}_1}$ . Also, by considering the operator  $R = \begin{pmatrix} R_0 \\ R_1 \end{pmatrix} : \mathcal{H}_2 \rightarrow \mathcal{H}_0 \oplus \mathcal{H}_1 =: \mathcal{M}$ , we can write  $T$  in the form

$$T = \begin{pmatrix} W & R \\ 0 & Q \end{pmatrix}$$

on  $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ . This representation gives

$$T^*T = \begin{pmatrix} W^*W & W^*R \\ R^*W & R^*R + Q^*Q \end{pmatrix},$$

and since  $\mathcal{H}_1 \subset \mathcal{N}(I - T^*T)$  one has  $W^*W|_{\mathcal{H}_1} = I_{\mathcal{H}_1}$ . This means

$$W^*W = \begin{pmatrix} 0 & 0 \\ 0 & W_0^*W_0 + W_1^*W_1 \end{pmatrix} = 0 \oplus I_{\mathcal{H}_1},$$

that is  $W$  is a partial isometry on  $\mathcal{M}$ . On the other hand, we have

$$TT^* = \begin{pmatrix} WW^* + RR^* & RQ^* \\ QR^* & QQ^* \end{pmatrix} \leq I,$$

whence it follows  $RR^* \leq I_{\mathcal{M}} - P_{\mathcal{R}(W)} = P_{\mathcal{N}(W^*)}$ , which means  $W^*R = 0$ .

To show that  $\mathcal{M}$  is the maximum invariant subspace for  $T$  in  $\mathcal{H}$  on which  $T$  is a partial isometry, let us consider another such subspace  $\mathcal{M}' \subset \mathcal{H}$ . So,  $T$  has a similar block matrix form on  $\mathcal{H} = \mathcal{M}' \oplus (\mathcal{M}')^\perp$ , namely

$$T = \begin{pmatrix} W' & R' \\ 0 & Q' \end{pmatrix}$$

with  $W'$  a partial isometry on  $\mathcal{M}'$  satisfying (as above)  $W'^*R' = 0$ . Then

$$T^*T = \begin{pmatrix} W'^*W' & 0 \\ 0 & R'^*R' + Q'^*Q' \end{pmatrix}$$

and this gives

$$\mathcal{N}(W') \subset \mathcal{N}(T) \subset \mathcal{M}.$$

As  $W'$  is a partial isometry we have the inclusion

$$\mathcal{R}(W'^*) = \mathcal{N}(I - W'^*W') \subset \mathcal{N}(I - T^*T) \cap \mathcal{M}'$$

and, in fact, the equality holds here. Indeed, if  $x = T^*Tx \in \mathcal{M}'$  and  $x \perp \mathcal{R}(W'^*)$  then  $0 = W'x = TT^*x$  so  $Tx = 0$  that is  $x = 0$ . Hence we get

$$\mathcal{M}' = \mathcal{N}(W') \oplus \mathcal{R}(W'^*) \subset \mathcal{N}(T) \oplus \mathcal{R}(W'^*),$$

which implies

$$T\mathcal{R}(W'^*) = W'\mathcal{R}(W'^*) \subset \mathcal{M}' \subset \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T) \cap \mathcal{M}'.$$

We infer (by using the operator  $T_2$  in (2.3)) that

$$T_2\mathcal{R}(W'^*) = P_{\mathcal{H}'}T|_{\mathcal{N}(I - T^*T) \cap \mathcal{M}'} = 0,$$

which means  $\mathcal{R}(W'^*) \subset \mathcal{N}(T_2) = \mathcal{H}_1$ . Conclude (by an above inclusion) that

$$\mathcal{M}' \subset \mathcal{N}(T) \oplus \mathcal{H}_1 = \mathcal{M},$$

that is  $\mathcal{M}$  has the required maximality property.

Now we prove the other properties concerning the operator  $Q$ . Firstly, if  $x \in \mathcal{N}(I - Q^*Q)$  then

$$\|x\| = \|Qx\| = \|P_{\mathcal{M}^\perp}Tx\| \leq \|Tx\| \leq \|x\|,$$

so  $x \in \mathcal{N}(I - T^*T)$ . This gives  $\mathcal{N}(I - Q^*Q) \subset \mathcal{N}(I - T^*T)$ .

Clearly, the equality (2.2) is equivalent to the following :

$$\mathcal{N}(I - T^*T) = \mathcal{H}_1 \oplus \mathcal{N}(I - Q^*Q).$$

Let  $x \in \mathcal{N}(I - T^*T) \cap \mathcal{M}^\perp$ , so  $x = T^*Tx$ . Then

$$Qx = P_{\mathcal{M}^\perp}TT^*Tx = P_{\mathcal{M}^\perp}Tx$$

and we have

$$\begin{aligned} \|x\|^2 = \|Tx\|^2 &= \|P_{\mathcal{M}}Tx\|^2 + \|P_{\mathcal{M}^\perp}Tx\|^2 \\ &= \|Rx\|^2 + \|Qx\|^2. \end{aligned}$$

Hence  $x \in \mathcal{N}(I - Q^*Q)$  if and only if  $Rx = 0$ . This proves the last assertion of theorem, having in view the inclusion of  $\mathcal{N}(I - Q^*Q)$  into  $\mathcal{N}(I - T^*T)$  already quoted.  $\square$

**Corollary 2.2.** *Let  $T$  be a contraction on  $\mathcal{H}$ . Then the subspace  $\mathcal{N}(T) \oplus \mathcal{N}(I - T^*T)$  is invariant for  $T$  if and only if  $Q$  (in (2.1)) is a pure contraction on  $\mathcal{M}^\perp$ .*

*Proof.* If  $\mathcal{N}(T) \oplus \mathcal{N}(I - T^*T)$  is invariant for  $T$  then it is reduced to  $\mathcal{M}$ , hence  $\mathcal{N}(I - Q^*Q) = \{0\}$  by Theorem 2.1. This means that the contraction  $Q$  is pure. Conversely, assuming  $Q$  pure, we have also  $Q^* = T^*|_{\mathcal{M}^\perp}$  pure. So,  $\mathcal{M}^\perp = \overline{\mathcal{R}(I - QQ^*)} \subset \overline{\mathcal{R}(I - TT^*)}$  which implies

$$T\mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*) \subset \mathcal{M} \subset \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T).$$

Hence  $\mathcal{N}(T) \oplus \mathcal{N}(I - T^*T)$  is invariant for  $T$ .  $\square$

**Remark 2.3.** Recall [14] that the maximum subspace invariant for a contraction  $T$  on which  $T$  is an isometry is  $\mathcal{N}(I - S_T)$ , where  $S_T$  is the asymptotic limit of  $T$ , defined as the strong limit of the powers  $T^{*n}T^n$ ,  $n \geq 1$ . Therefore  $\mathcal{N}(I - S_T) \subset \mathcal{M} \cap \mathcal{N}(I - T^*T)$  and, in general, the inclusion is strict, because the powers  $T^n$  are not always partial isometries on  $\mathcal{M}$ .

In the case that  $\mathcal{M} = \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T)$  and  $\mathcal{N}(T^*)$  is invariant for  $T$  (that is  $\mathcal{N}(T^*) \subset \mathcal{N}(T)$ ), and if  $\mathcal{M}^* \subset \mathcal{H}$  is the corresponding subspace for  $T^*$  given by Theorem 2.1, then  $\mathcal{M}^* \subset \mathcal{N}(T^*) \oplus \mathcal{N}(I - TT^*) = \mathcal{N}(T^*) \oplus T\mathcal{N}(I - T^*T) \subset \mathcal{M}$ . In addition, if  $\mathcal{N}(I - TT^*)$  is invariant for  $T$  then

$$T\mathcal{M}^* = \mathcal{N}(T^*) \oplus T\mathcal{N}(I - TT^*) \subset \mathcal{N}(T^*) \oplus \mathcal{N}(I - TT^*) = \mathcal{M}^*,$$

hence  $\mathcal{M}^*$  reduces  $T$  to a partial isometry.

Clearly, the maximum subspace which reduces  $T$  to a partial isometry exists always, but it is different of  $\mathcal{M} \cap \mathcal{M}^*$ , in general. Its structure is more complicated, and will not be given here.

In the following section we see that the subspace  $\mathcal{M}$  has the form from Corollary 2.2 with  $\mathcal{N}(I - T^*T)$  reducing for  $T$ , under the condition (1.1), but  $\mathcal{M}$  does not reduce  $T$ , in general.

### 3. ON THE FONG-TSUI CONJECTURE

Remark firstly that in the Fong-Tsui conjecture one can suppose that  $T$  is a contraction, because the condition (1.1) works simultaneously for  $T$  and  $T|_{\|T\|}$ .

Concerning the structure of such a contraction we have the following main result.

**Theorem 3.1.** *Let  $T$  be a contraction on  $\mathcal{H}$  satisfying the condition (1.1). Then*

$$(3.1) \quad \mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*) = \mathcal{N}(I - |\operatorname{Re}T|)$$

*and this subspace reduces  $T$  to a symmetry. Also, we have*

$$(3.2) \quad \mathcal{N}(\operatorname{Re}T) = \mathcal{N}(T) \cap \mathcal{N}(T^*), \quad \mathcal{N}(T^*) = \mathcal{N}(\operatorname{Re}T) \oplus \mathcal{N}(T^*|_{\overline{\mathcal{R}(T^*)}}).$$

*Moreover, one has  $\mathcal{N}(T) = \mathcal{N}(\operatorname{Re}T)$  if and only if  $T = U \oplus Z$  with respect to a decomposition  $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$ , where  $U$  is a symmetry, and  $Z$  is a pure contraction satisfying the condition (1.1).*

*Proof.* Consider the block matrix (2.4) of  $T$  that is

$$T = \begin{pmatrix} 0 & W_0 & R_0 \\ 0 & W_1 & R_1 \\ 0 & 0 & Q \end{pmatrix}$$

on  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ , where  $\mathcal{H}_0 := \mathcal{N}(T)$ ,  $\mathcal{H}_1 := \mathcal{N}(P'T|_{\mathcal{N}(I-T^*T)})$ , while  $P'$  is the orthogonal projection onto  $\mathcal{H}' := \overline{\mathcal{R}(T^*T - (T^*T)^2)}$ , and  $\mathcal{H}_2 = [\mathcal{N}(I - T^*T) \ominus \mathcal{H}_1] \oplus \mathcal{H}'$ . In addition, from the proof of Theorem 2.1 we have the relations

$$(3.3) \quad W_0^*W_0 + W_1^*W_1 = I, \quad W_0^*R_0 + W_1^*R_1 = 0,$$

and

$$(3.4) \quad W_0W_0^* + R_0R_0^* \leq I, \quad W_1W_1^* + R_1R_1^* \leq I.$$

By a simple computation we get the representations :

$$T^*T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & R_0^*R_0 + R_1^*R_1 + Q^*Q \end{pmatrix}, \quad \text{Re}T = \frac{1}{2} \begin{pmatrix} 0 & W_0 & R_0 \\ W_0^* & 2\text{Re}W_1 & R_1 \\ R_0^* & R_1^* & 2\text{Re}Q \end{pmatrix}$$

and respectively (by using the second relation of (3.3))

$$(\text{Re}T)^2 = \frac{1}{4} \begin{pmatrix} W_0W_0^* + R_0R_0^* & 2W_0\text{Re}W_1 + R_0R_1^* & W_0R_1 + 2R_0\text{Re}Q \\ 2(\text{Re}W_1)W_0^* + R_1R_0^* & W_0^*W_0 + 4(\text{Re}W_1)^2 + R_1R_1^* & W_1R_1 + 2R_1\text{Re}Q \\ R_1^*W_0^* + 2(\text{Re}Q)R_0^* & R_1^*W_1^* + 2(\text{Re}Q)R_1^* & R_0^*R_0 + R_1^*R_1 + 4(\text{Re}Q)^2 \end{pmatrix}.$$

Since the subspace  $\mathcal{H}_1$  reduces  $|T|$  to  $I_{\mathcal{H}_1}$  and as  $\text{Re}T$  is a contraction, the condition (1.1) implies that  $\mathcal{H}_1$  also reduces  $|\text{Re}T|$  and  $|\text{Re}T|_{\mathcal{H}_1} = I_{\mathcal{H}_1}$ . As  $|\text{Re}T|^2 = (\text{Re}T)^2$  we get from the above matrix representation that

$$(3.5) \quad \frac{1}{4}(W_0^*W_0 + 4(\text{Re}W_1)^2 + R_1R_1^*) = I_{\mathcal{H}_1}.$$

By using the former relation in (3.3) and the second relation in (3.4) we obtain

$$I_{\mathcal{H}_1} \leq \text{Re}W_1^2,$$

which means by [11, Corollary 3] that  $W_1^2 = I$ . So, the relation (3.5) becomes  $W_1W_1^* + R_1R_1^* = I_{\mathcal{H}_1}$  which leads to  $I_{\mathcal{H}_1} + W_1R_1R_1^*W_1^* = W_1W_1^*$ . But this gives  $R_1 = 0$  and  $W_1W_1^* = I_{\mathcal{H}_1}$ , hence  $W_1 = W_1^*$ , while by (3.3) this yields  $W_0 = 0$ . So, the block matrices of  $T$ ,  $T^*T$ ,  $\text{Re}T$  and  $(\text{Re}T)^2$  have simpler forms.

Now, we have by Theorem 2.1 that  $\mathcal{N}(I - Q^*Q) \subset \mathcal{N}(I - T^*T) \cap \mathcal{H}_2$ , and we show next that  $\mathcal{N}(I - T^*T) \cap \mathcal{H}_2 \subset \mathcal{N}(I - QQ^*)$ . Indeed, let  $x = T^*Tx \in \mathcal{H}_2$ . To use the condition (1.1)

we consider  $|\operatorname{Re}T|$  to have the following block matrix form on  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$  (having in view that  $|\operatorname{Re}T|_{|\mathcal{H}_1} = I_{\mathcal{H}_1}$ ) :

$$|\operatorname{Re}T| = \begin{pmatrix} A & 0 & B \\ 0 & I & 0 \\ B^* & 0 & C \end{pmatrix},$$

with some appropriate contractions  $A, B$  and  $C$  with  $A, C \geq 0$ . As  $|\operatorname{Re}T|^2 = (\operatorname{Re}T)^2$  we obtain that  $B^*B + C^2 = \frac{1}{4}[R_0^*R_0 + (Q + Q^*)^2]$ , so for  $x$  as above we get (by (1.1))

$$\|x\|^2 = \langle |T|x, x \rangle \leq \langle |\operatorname{Re}T|x, x \rangle = \langle Cx, x \rangle \leq \|x\|^2$$

which means  $Cx = x$  (because  $C \geq 0$ ). Then the above equality together with the fact that  $(R_0^*R_0 + Q^*Q)x = T^*Tx = x$ , lead to the relation

$$\frac{3}{4}\|x\|^2 + \|Bx\|^2 = \frac{1}{4}\|Q^*x\|^2 + \frac{1}{2}\langle \operatorname{Re}Q^2x, x \rangle.$$

Since  $Q^2$  is a contraction it follows that

$$\frac{1}{4}(\|x\|^2 - \|Q^*x\|^2) + \|Bx\|^2 \leq 0,$$

hence  $Bx = 0$  and  $\|Q^*x\| = \|x\|$ . So  $x \in \mathcal{N}(I - QQ^*)$  and the inclusion  $\mathcal{N}(I - T^*T) \cap \mathcal{H}_2 \subset \mathcal{N}(I - QQ^*)$  is proved.

Next, we consider the block matrix form of  $TT^*$  on  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ , namely

$$TT^* = \begin{pmatrix} R_0R_0^* & 0 & R_0Q^* \\ 0 & I & 0 \\ QR_0^* & 0 & QQ^* \end{pmatrix}.$$

Since  $T^*|_{\mathcal{H}_2} = Q^*$  and  $T$  is a contraction we have  $\mathcal{N}(I - QQ^*) \subset \mathcal{N}(I - TT^*)$ , and from this representation of  $TT^*$  one has also  $\mathcal{H}_1 \subset \mathcal{N}(I - TT^*)$ . So we infer that

$$(3.6) \quad \mathcal{N}(I - T^*T) = \mathcal{H}_1 \oplus \mathcal{N}(I - T^*T) \cap \mathcal{H}_2 \subset \mathcal{H}_1 \oplus \mathcal{N}(I - QQ^*) \subset \mathcal{N}(I - TT^*).$$

This means that  $\mathcal{N}(I - TT^*)$  is invariant for  $T^*$ , hence  $\mathcal{N}(I - T^*T)$  is also invariant for  $T^*$ .

To see that  $\mathcal{N}(I - T^*T)$  just reduces  $T$  we firstly remark from (1.1) that

$$\mathcal{N}(I - T^*T) = \mathcal{N}(I - |T|) \subset \mathcal{N}(I - |\operatorname{Re}T|) = \mathcal{N}(I - (\operatorname{Re}T)^2).$$

In fact, these subspaces coincide, they containing the subspace  $\mathcal{H}_1$ . To see this equality, let us consider the polar decomposition

$$\operatorname{Re}T = \tilde{U}|\operatorname{Re}T|,$$

where  $\tilde{U}$  is a symmetry on  $\overline{\mathcal{R}(\operatorname{Re}T)} = \overline{\mathcal{R}(|\operatorname{Re}T|)}$ . Clearly, one has

$$\mathcal{N}(I - |\operatorname{Re}T|) = \mathcal{N}(\tilde{U} - \operatorname{Re}T)$$

and this subspace reduces the operators  $\operatorname{Re}T$  and  $\tilde{U}$ .



Let  $x \in \mathcal{N}(\tilde{U} - \text{Re}T) \cap \mathcal{H}_2$  such that  $x$  is orthogonal on  $\mathcal{N}(I - T^*T)$ , hence  $\|Tx\| < \|x\|$ . As  $\mathcal{H}_2 \subset \overline{\mathcal{R}(\text{Re}T)}$  and  $\tilde{U}$  is unitary on this range, we get

$$\|x\| = \|Ux\| = \|(\text{Re}T)x\| \leq \frac{1}{2}(\|Tx\| + \|T^*x\|) < \|x\|$$

which forces to have  $x = 0$ . Therefore  $\mathcal{N}(I - |\text{Re}T|) \cap \mathcal{H}_2 \subset \mathcal{N}(I - T^*T)$ , and since  $\mathcal{H}_1 \subset \mathcal{N}(I - |\text{Re}T|)$  we conclude that  $\mathcal{N}(I - T^*T) = \mathcal{N}(I - |\text{Re}T|)$ . But this subspace reduces  $\text{Re}T$  and it is invariant for  $T^*$  (as we have seen before). Hence  $\mathcal{N}(I - T^*T)$  reduces  $T$ , which also gives the inclusion  $\mathcal{N}(I - TT^*) \subset \mathcal{N}(I - T^*T)$ . Finally, we conclude that

$$\mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*) = \mathcal{N}(I - |\text{Re}T|) = \mathcal{H}_1,$$

where for the last equality we have in view the maximality of the subspace  $\mathcal{N}(T) \oplus \mathcal{H}_1$  relative to  $T$ . The identities (3.1) are proved.

In addition, from the inclusions (3.6) we infer that  $\mathcal{N}(I - QQ^*) = \{0\}$ , so  $Q^*$  like  $Q$  are pure contractions on  $\mathcal{H}_2$ .

Now, the condition (1.1) yields  $\mathcal{N}(\text{Re}T) = \mathcal{N}(T) \cap \mathcal{N}(T^*)$ , and this subspace reduces  $T$ . Therefore we have  $\mathcal{N}(\text{Re}T) = \mathcal{N}(T)$  if and only if  $\mathcal{N}(T)$  reduces  $T$ , that is  $R_0 = 0$ . Equivalently, this means that  $T$  has the diagonal representation  $T = W_1 \oplus 0 \oplus Q$  on  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_2$ , with  $W_1, Q$  as above, that is  $T = W_1 \oplus Z$  where  $Z = 0 \oplus Q$  is a pure contraction which satisfies the condition (1.1).

Finally, for the second relation in (3.2) we have from the above block matrix form of  $T$  (with  $W_0 = 0, R_1 = 0$ )

$$\mathcal{N}(T^*) = \{y \oplus z \in \mathcal{N}(T) \oplus \mathcal{H}_2 : R_0^*y + Q^*z = 0\}.$$

But  $\mathcal{N}(T^*) \cap \mathcal{N}(T) = \mathcal{N}(R_0^*)$  and  $\mathcal{N}(T^*|_{\mathcal{H}_2}) = \mathcal{N}(Q^*) = \mathcal{N}(T^*) \cap \mathcal{H}_2 = \mathcal{N}(T^*|_{\overline{\mathcal{R}(T^*)}})$ . So we obtain

$$\mathcal{N}(T^*) = \mathcal{N}(R_0^*) \oplus \mathcal{N}(Q^*) = \mathcal{N}(\text{Re}T) \oplus \mathcal{N}(T^*|_{\overline{\mathcal{R}(T^*)}}),$$

and this finishes the proof.  $\square$

**Corollary 3.2.** *A real scalar multiple of a partial isometry which satisfies the condition (1.1) is self-adjoint.*

*Proof.* If  $T$  is a partial isometry satisfying (1.1) then  $\mathcal{H} = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T)$ , so  $T = U \oplus 0 = T^*$  by Theorem 3.1. More general, if  $T = \alpha T_0$  with  $\alpha \in \mathbb{R}$  and  $T_0$  a partial isometry, then  $\frac{1}{\alpha}T = \frac{1}{\alpha}T^*$  by the previous remark, so  $T = T^*$ .  $\square$

A more general result than the previous corollary is the following

**Proposition 3.3.** *Let  $0 \neq T \in \mathcal{B}(\mathcal{H})$  having with respect to a decomposition  $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$  the block matrix form*

$$(3.7) \quad T = \begin{pmatrix} S & R \\ 0 & Q \end{pmatrix},$$

where  $\frac{1}{\|T\|}S$  is an isometry on  $\mathcal{G}$  and, in addition, either  $R \in \mathcal{B}(\mathcal{G}^\perp, \mathcal{G})$  is injective and  $Q \in \mathcal{B}(\mathcal{G}^\perp)$  is arbitrary, or  $Q^2 = 0$ . If  $T$  satisfies the condition (1.1) then  $T$  is self-adjoint.

*Proof.* Let  $T \neq 0$  as in (3.7), and let  $T_0 = \frac{1}{\alpha}T$ ,  $S_0 = \frac{1}{\alpha}S$ ,  $R_0 = \frac{1}{\alpha}R$ ,  $Q_0 = \frac{1}{\alpha}Q$ , where  $\alpha = \|T\|$ . Then the contraction  $T_0$  satisfies (1.1), and by Theorem 3.1 we infer  $\mathcal{N}(I - T_0^*T_0) = \mathcal{N}(I - T_0T_0^*)$ . To use this fact we have in view the representations

$$I - T_0^*T_0 = \begin{pmatrix} 0 & 0 \\ 0 & I - R_0^*R_0 - Q_0^*Q_0 \end{pmatrix}, \quad I - T_0T_0^* = \begin{pmatrix} I - S_0S_0^* - R_0R_0^* & -R_0Q_0^* \\ -Q_0R_0^* & I - Q_0Q_0^* \end{pmatrix},$$

where for  $I - T_0^*T_0$  we used that  $S_0$  is an isometry on  $\mathcal{G}$ . These representations together with the above kernels give that  $\mathcal{G} \subset \mathcal{N}(I - T_0T_0^*)$ , that is the relations

$$(3.8) \quad I - S_0S_0^* - R_0R_0^* = 0, \quad Q_0R_0^* = 0.$$

From the first relation we have  $R_0R_0^* = P_{\mathcal{N}(S_0^*)}$ , so  $R_0$  is a partial isometry and  $\mathcal{R}(R_0) = \mathcal{N}(S_0^*)$ .

Now, if  $R$  is injective, then from the second relation in (3.8) one obtains  $Q_0 = 0$ . In this case  $R_0$  is an isometry on  $\mathcal{G}^\perp$ , and since by the above matrix representations we get

$$\mathcal{G} = \mathcal{N}(I - T_0T_0^*) = \mathcal{N}(I - T_0^*T_0) = \mathcal{G} \oplus \mathcal{N}(I - R_0^*R_0) = \mathcal{G} \oplus \mathcal{G}^\perp,$$

it follows  $\mathcal{G}^\perp = \{0\}$ . Hence  $T_0 = S_0$  is a symmetry on  $\mathcal{H}$ , and consequently  $T = T^*$ .

Assume next the other condition from hypothesis, namely for  $Q^2 = 0$ . Since  $\mathcal{N}(I - T_0^*T_0) = \mathcal{G} \oplus \mathcal{N}(I - Q_0Q_0^*)$  is invariant for  $T_0^*$ ,  $\mathcal{N}(I - Q_0Q_0^*)$  will be invariant for  $Q_0^* = T_0^*|_{\mathcal{G}^\perp}$ . So, if  $b \in \mathcal{N}(I - Q_0Q_0^*)$  we have  $Q_0^*b = Q_0Q_0^{*2}b = 0$ , which means  $b = Q_0Q_0^*b \in \mathcal{N}(Q_0^*)$  that is  $b = 0$ . Hence  $\mathcal{N}(I - Q_0Q_0^*) = \{0\}$  which gives  $\mathcal{N}(I - T_0T_0^*) = \mathcal{G}$  and this subspace reduces  $T_0$ . Thus  $R_0 = 0$ , while  $S_0$  and  $Q_0$  satisfy the condition (1.1). Therefore  $S_0$  will be a symmetry on  $\mathcal{G}$ , and as  $Q_0^2 = 0$  it is easy to see that  $Q = 0$ . Indeed, since  $Q^2 = 0$ ,  $Q$  will have on  $\mathcal{G}^\perp = \overline{\mathcal{R}(Q)} \oplus \mathcal{N}(Q^*)$  the block matrix form

$$Q = \begin{pmatrix} 0 & Q_1 \\ 0 & 0 \end{pmatrix}.$$

By a simple computation we get

$$|Q| = 0 \oplus |Q_1|, \quad |\operatorname{Re} Q| = \frac{1}{2}(|Q_1^*| \oplus |Q_1|)$$

on the above decomposition of  $\mathcal{G}^\perp$ . So, the condition (1.1) for  $Q$  implies  $Q_1 = 0$  that is  $Q = 0$ .

We conclude that  $T = S \oplus 0 = \alpha S_0 \oplus 0 = T^*$ ,  $S_0$  being a symmetry. This ends the proof.  $\square$

To apply this proposition for 2-quasi-isometries, we recall (see [8, Remark 3.10]; or [15, Remark 2.7]) that such an operator has a block matrix form as in (3.7) on  $\mathcal{H} = \overline{\mathcal{R}(T^2)} \oplus \mathcal{N}(T^{*2})$ , with  $S$  an isometry and  $Q^2 = 0$ . So the previous proposition gives the following

**Corollary 3.4.** *A contractive 2-quasi-isometry which satisfies the condition (1.1) is self-adjoint.*

It is still unknown if the second assumption in the hypothesis of Proposition 3.3, namely  $Q^2 = 0$ , can be replaced by the weaker condition  $Q^m = 0$  for some  $m \geq 3$ , in order to preserve the conclusion; that is to prove that  $Q = 0$  under the condition (1.1).

However, for  $m$ -quasi-isometries we have the following

**Corollary 3.5.** *If  $T \in \mathcal{B}(\mathcal{H})$  is a contractive  $m$ -quasi-isometry for an integer  $m \geq 3$  which satisfies the condition (1.1) then  $\mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*) = \overline{\mathcal{R}(T^m)}$  reduces  $T$  to a symmetry. In addition,  $T$  is self-adjoint if and only if  $T = S \oplus 0$ .*

*Proof.* If  $T$  is a  $m$ -quasi-isometry then  $T|_{\mathcal{R}(T^m)}$  is an isometry. Such an operator has the form (3.7) on  $\mathcal{H} = \overline{\mathcal{R}(T^m)} \oplus \mathcal{N}(T^{*m})$  with  $S$  an isometry and  $Q^m = 0$ . In the case when  $\|T\| = 1$  we also have (as in the proof of Proposition 3.3) that  $S^*R = 0$  and

$$\mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*) = \overline{\mathcal{R}(T^m)} \oplus \mathcal{N}(I - QQ^*),$$

this subspace reducing  $T$  to a symmetry. So, if  $b \in \mathcal{N}(I - QQ^*)$  and  $m \geq 3$  then  $Q^*b \in \mathcal{N}(I - QQ^*)$  which leads (by recurrence) to  $b = QQ^*b = Q^m Q^{*m}b = 0$ . Hence  $\mathcal{N}(I - QQ^*) = \{0\}$ ,  $\mathcal{N}(I - T^*T) = \overline{\mathcal{R}(T^m)}$ , and  $S$  is a symmetry on this subspace. Then the condition  $S^*R = 0$  yields  $R = 0$ , consequently  $T = S \oplus Q$ .

Assume now  $T = T^*$  that is  $Q = Q^*$ . Since  $Q^m = 0$  one has  $\mathcal{R}(Q^{m-1}) \subset \mathcal{N}(Q) = \mathcal{N}(Q^*) \subset \mathcal{N}(Q^{*(m-1)})$ , hence  $Q^{m-1} = 0$ . By recurrence one infers  $Q = 0$ , so  $T = S \oplus 0$ . The converse implication for the second assertion of corollary being trivial, the proof is finished.  $\square$

We remarked before that a non-null nilpotent operator of order 2 cannot satisfy the condition (1.1). We can also use this fact to obtain the following result

**Proposition 3.6.** *Let  $T$  be a contraction on  $\mathcal{H}$  having with respect to the decomposition  $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$  the block matrix form*

$$(3.9) \quad T = \begin{pmatrix} W & R \\ 0 & W' \end{pmatrix}$$

*where  $W$  and  $W'$  are partial isometries on  $\mathcal{G}$  and  $\mathcal{G}^\perp$ , respectively, while  $R \in \mathcal{B}(\mathcal{G}^\perp, \mathcal{G})$ . If  $T$  satisfies the condition (1.1) then  $T$  is self-adjoint.*

*Proof.* Consider firstly that  $W' = 0$  in (3.9). Then  $\overline{\mathcal{R}(T)} \subset \mathcal{G}$  and, assuming (1.1), by Theorem 2.1 and Theorem 3.1 one has  $\mathcal{G} \subset \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T) =: \mathcal{M}$ . Therefore  $\mathcal{M}^\perp \subset \mathcal{G}^\perp \subset \mathcal{N}(T^*)$  that is  $T^*|_{\mathcal{M}^\perp} = 0$ , and using the  $3 \times 3$  block matrix of  $T$  on  $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(I - T^*T) \oplus \mathcal{M}^\perp$  given by the proof of Theorem 3.1 (with  $W_0 = 0$ ,  $R_1 = 0$ ,  $Q = 0$  by the previous remark,  $W_1$  a symmetry on  $\mathcal{N}(I - T^*T)$ , and  $R_0 \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{N}(T))$ ) we get the representation

$T = W_1 \oplus \tilde{R}$  on  $\mathcal{H} = \mathcal{N}(I - T^*T) \oplus [\mathcal{N}(T) \oplus \mathcal{M}^\perp]$ , where  $\tilde{R} \in \mathcal{B}(\mathcal{N}(T) \oplus \mathcal{M}^\perp)$  has the block matrix form

$$\tilde{R} = \begin{pmatrix} 0 & R_0 \\ 0 & 0 \end{pmatrix}.$$

Since  $\tilde{R}^2 = 0$  and  $\tilde{R}$  satisfies (1.1) one has  $\tilde{R} = 0$ , which ensures  $T = T^*$ .

In the general case, as  $W'$  is a partial isometry on  $\mathcal{G}^\perp$  and  $\mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*)$  by (3.1), we have  $\mathcal{G}^\perp \subset \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T^*)$ , and as above  $\mathcal{G} \subset \mathcal{M}$ . So,  $\mathcal{M}^\perp \subset \mathcal{G}^\perp$  which by the previous inclusion of  $\mathcal{G}^\perp$  gives  $\mathcal{M}^\perp \subset \mathcal{N}(T^*)$ . This shows that  $T$  has on  $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$  a block matrix of the form

$$T = \begin{pmatrix} \widetilde{W} & \tilde{R} \\ 0 & 0 \end{pmatrix},$$

with  $\widetilde{W} = 0 \oplus W_1$  a partial isometry and then it follows that  $T = T^*$  by the previous conclusion.  $\square$

The contractions mentioned in this proposition are not hyponormal, in general. But among these one gets the quasi-isometries (the case when  $W$  is an isometry), which are subnormal (as we already quoted in the introduction).

**Remark 3.7.** An operator  $T \in \mathcal{B}(\mathcal{H})$  having the block matrix form (3.7) as in Proposition 3.3 is not a contraction, in general. But in the case when  $T$  is a contraction with  $S$  an isometry and  $R, Q$  arbitrary contractions (in (3.7)), then the condition  $S^*R = 0$  is also true and  $R$  will be a partial isometry (as in the proof of Proposition 3.3), if  $T$  satisfies (1.1). In general  $R \neq 0$ , but if  $Q$  is pure then  $R = 0$  because  $\mathcal{N}(I - T^*T)$  reduces  $T$  to a symmetry and we have

$$\mathcal{N}(I - T^*T) = \mathcal{N}(I - TT^*) = \mathcal{G} \oplus \mathcal{N}(I - R^*R - Q^*Q) = \mathcal{G} \oplus \mathcal{N}(I - QQ^*).$$

This latter case occurs, for instance, when  $\mathcal{G} = \mathcal{N}(I - S_T)$  in the representation (3.7) of  $T$ , and then the Fong-Tsui conjecture for  $T$  one reduces to its pure part  $Q$ .

In fact, every operator  $T \neq 0$  on  $\mathcal{H}$  has the form (3.7) with  $R, Q$  arbitrary operators, on the decomposition  $\mathcal{H} = \mathcal{N}(I - S_{T_0}) \oplus \overline{\mathcal{R}(I - S_{T_0})}$ , where  $T_0 = \frac{1}{\|T\|}T$ ,  $S_{T_0}$  is the asymptotic limit of  $T_0$ , and so  $\frac{1}{\|T\|}S$  is an isometry (in (3.7)). Thus, the assumptions on the operators  $R$  or  $Q$  in Proposition 3.3 and the condition (1.1) force such operator  $T$  to be self-adjoint.

In turn to Proposition 3.6, it is clear that in (3.9) one can consider multiples of partial isometries with the scalar  $\alpha = \|T\|$  instead of  $W$  and  $W'$  respectively, in order to preserve the conclusion.

Another special class of non-contractive operators which contains some 2-isometries as well as the Brownian isometries, and for which the Fong-Tsui conjecture holds, is mentioned by the following

**Proposition 3.8.** *Let  $T \in \mathcal{B}(\mathcal{H})$  such that  $T^*T \geq I$  and having the block matrix form (3.7) on  $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$ , where  $S$  is an isometry on  $\mathcal{G}$ ,  $R \in \mathcal{B}(\mathcal{G}^\perp, \mathcal{G})$  with  $S^*R = 0$ , while  $Q$  is a contraction on  $\mathcal{G}^\perp$ . If  $T$  satisfies the condition (1.1) then  $T$  is self-adjoint, in fact a symmetry.*

*Proof.* By using (3.7) with  $S, R, Q$  as above one obtains

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & R^*R + Q^*Q \end{pmatrix}, \quad (\operatorname{Re}T)^2 = \frac{1}{4} \begin{pmatrix} (S + S^*)^2 + RR^* & SR + R(Q + Q^*) \\ R^*S^* + (Q + Q^*)R^* & R^*R + (Q + Q^*)^2 \end{pmatrix}.$$

Since  $T^*T \geq I$  the condition  $|T| \leq |\operatorname{Re}T|$  implies  $|\operatorname{Re}T| \geq I$ . Therefore we have  $|T| \leq |\operatorname{Re}T| \leq (\operatorname{Re}T)^2$  which gives for any  $x \in \mathcal{G}$  (as  $R^*S = 0$ ),

$$\begin{aligned} \|x\|^2 &= \|Sx\|^2 = \langle |T|Sx, Sx \rangle \leq \langle (\operatorname{Re}T)^2Sx, Sx \rangle \\ &= \frac{1}{4} \langle ((S + S^*)^2 + RR^*)Sx, Sx \rangle = \frac{1}{4} \langle (S^3 + S^* + 2S)x, Sx \rangle \\ &= \frac{1}{2} (\|x\|^2 + \langle (\operatorname{Re}S^2)x, x \rangle). \end{aligned}$$

This means that  $I_{\mathcal{G}} \leq \operatorname{Re}S^2$ , which by [10, Corollary 3] yields  $S^2 = I_{\mathcal{G}}$ . As  $S$  is an isometry it will be just a symmetry, and as  $S^*R = 0$  it follows  $R = 0$ .

On the other hand, since  $I \leq (\operatorname{Re}T)^2$  we have for each  $y \in \mathcal{G}^\perp$  ( $Q$  being a contraction)

$$\|y\|^2 \leq \frac{1}{4} \langle (Q + Q^*)^2 y, y \rangle \leq \frac{1}{4} \langle (Q^2 + Q^{*2} + 2I)y, y \rangle,$$

whence  $I_{\mathcal{G}^\perp} \leq \operatorname{Re}Q^2$ . Then as above  $Q$  is a symmetry, hence  $T = S \oplus Q = T^*$ . This ends the proof.  $\square$

In concordance with the representations (1.2) and (3.7) for Brownian isometries, from the previous proposition we derive the following

**Corollary 3.9.** *A Brownian isometry of positive covariance which satisfies the condition (1.1) is a symmetry.*

**Remark 3.10.** Proposition 3.8 one refers to a larger class of operators than that of Brownian isometries, but not to all 2-isometries, because we need to impose that  $Q$  is a contraction.

Recall (see [1, Theorem 1.26]) that the block matrix of a 2-isometry on  $\mathcal{H} = \mathcal{N}(T^*T - I) \oplus \overline{\mathcal{R}(T^*T - I)}$  has the form (3.7), with  $S$  an isometry,  $S^*R = 0$ ,  $R^*R + Q^*Q - I$  injective and

$$Q^*(R^*R + Q^*Q - I)Q = R^*R + Q^*Q - I.$$

But if such  $T$  satisfies the condition (1.1), then as in the previous proof  $S$  will be a symmetry, hence  $R = 0$ . In this case, the last relation before means  $Q^*(Q^*Q - I)Q = Q^*Q - I$ , that is  $Q$  is a 2-isometry and  $Q^*Q \geq I$ . So, the Fong-Tsui conjecture for  $T$  one reduces to its 2-isometric pure part  $Q$ .

## 4. FINAL REMARKS

In turn to Theorem 3.1 which plays an essential role for our considerations concerning the Fong-Tsui conjecture, we make some comments.

**Remark 4.1.** The operator  $T^*$  has a similar form like  $T$ , under the condition (1.1), and in the corresponding decomposition of  $\mathcal{H}$ . Indeed, we have from the proof of Theorem 3.1 the representation

$$(4.1) \quad T = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & 0 & R_0 \\ 0 & 0 & Q \end{pmatrix}$$

on the decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_2$  where  $\mathcal{H}_1 = \mathcal{N}(I - T^*T)$  and  $\mathcal{H}_0 = \mathcal{N}(T)$ . To see here  $\mathcal{N}(T^*)$  given by (3.2), we refine (4.1) by considering  $\mathcal{N}(T) = \mathcal{N}(\operatorname{Re}T) \oplus \mathcal{H}'_0$  and  $\mathcal{H}_2 = \mathcal{N}(Q^*) \oplus \overline{\mathcal{R}(Q)}$ . So, on the decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{N}(\operatorname{Re}T) \oplus \mathcal{N}(Q^*) \oplus \mathcal{H}'_0 \oplus \overline{\mathcal{R}(Q)}$  we have the representations

$$T = \begin{pmatrix} W_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{00} & 0 & R_{01} \\ 0 & 0 & Q_0 & 0 & Q_1 \end{pmatrix}, \quad T^* = \begin{pmatrix} W_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{00}^* & Q_0^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{01}^* & Q_1^* \end{pmatrix},$$

where  $R_{00} = P_{\mathcal{H}'_0} R_0|_{\mathcal{N}(Q^*)}$ ,  $R_{01} = P_{\mathcal{H}'_0} R_0|_{\overline{\mathcal{R}(Q)}}$ , and  $Q_0 = Q|_{\mathcal{N}(Q^*)}$ ,  $Q_1 = Q|_{\overline{\mathcal{R}(Q)}}$ . Here we used the fact that  $P_{\mathcal{N}(\operatorname{Re}T)} R_0 = 0$  and  $P_{\mathcal{N}(Q^*)} Q = 0$ . We get that  $T^*$  has on  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{N}(T^*) \oplus \mathcal{H}_3$  with  $\mathcal{H}_3 = \mathcal{H}'_0 \oplus \mathcal{H}_3$  the block matrix form

$$T^* = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & 0 & R_* \\ 0 & 0 & Q_* \end{pmatrix},$$

where  $R_* := \begin{pmatrix} 0 & 0 \\ R_{00}^* & Q_0^* \end{pmatrix}$  and  $Q_* := \begin{pmatrix} 0 & 0 \\ R_{01}^* & Q_1^* \end{pmatrix}$ .

Hence  $T^*$  has the same form as  $T$  in (4.1) on the corresponding decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{N}(T^*) \oplus \mathcal{H}_3$ , which was obtained under the condition (1.1), even if  $T^*$  does not satisfy this condition.

These representations of  $T$  and  $T^*$  also provide that the invariant partial isometric parts for  $T$  and  $T^*$  can be different under the condition (1.1), that is  $\mathcal{N}(T) \neq \mathcal{N}(T^*)$ , in general. But the condition  $\mathcal{N}(T) = \mathcal{N}(T^*)$  is necessary for  $T$  to be self-adjoint. In this latter case we have  $R_0 = 0$  and  $Q_0 = 0$ , while  $Q = Q_1$  like  $Q^*$  are injective and pure contractions, hence

$$\overline{\mathcal{R}(Q)} = \overline{\mathcal{R}(Q^*)} = \overline{\mathcal{R}(I - Q^*Q)} = \overline{\mathcal{R}(I - QQ^*)}.$$

But only these information on  $Q$  are not sufficient to obtain that  $Q = Q^*$  that is  $T = T^*$ .

The major difficulty to use the condition (1.1) in order to obtain the self-adjointness of  $T$  consists in the fact that  $|\operatorname{Re}T| = [(\operatorname{Re}T)^2]^{1/2}$  cannot be easily expressed in terms of  $T$  (that is using the block matrix form (4.1)).

Clearly, one can have in view the polar decomposition  $\operatorname{Re}T = U|\operatorname{Re}T|$ , where  $U$  is a symmetry on  $\overline{\mathcal{R}(\operatorname{Re}T)}$ . Since  $\mathcal{N}(\operatorname{Re}T) = \mathcal{N}(U)$  reduces  $U$  and  $T$ , one can just only consider the condition (1.1) on  $\overline{\mathcal{R}(\operatorname{Re}T)}$  which reduces  $T$  and  $U$ .

But even in the case when  $\mathcal{N}(T)$  reduces  $T$  ( $R_0 = 0$  in (3.9)), hence when  $U$  is a symmetry on the subspace  $\mathcal{H}_2$ , we cannot use this to obtain  $Q = Q^*$ .

But, as we have seen before, this is possible for some classes of operators related to partial isometries, or for the operators  $T$  which commute with  $U$ , as it was recently shown in [17]. Let us remark that this last class of operators and that of partial isometries are not contained one into the other. For example, one can consider the partial isometry  $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$  and the corresponding partial isometry for  $\operatorname{Re}T$  with the block matrices

$$T = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

and it is clear that  $TU = I \oplus 0$  and  $UT = 0 \oplus I$ .

**Remark 4.2.** The condition (1.1) means that there exists a contraction  $A$  on  $\mathcal{H}$  satisfying the relation

$$(4.2) \quad A|\operatorname{Re}T|^{1/2} = |T|^{1/2},$$

or equivalently

$$(4.3) \quad |\operatorname{Re}T|^{1/2}A^* = |T|^{1/2}.$$

From these relations we have  $\mathcal{N}(A^*) \subset \mathcal{N}(T)$  and also

$$A\overline{\mathcal{R}(\operatorname{Re}T)} \subset \overline{\mathcal{R}(T^*)} \subset \overline{\mathcal{R}(\operatorname{Re}T)}$$

that is  $\overline{\mathcal{R}(\operatorname{Re}T)}$  is invariant for  $A$ . Denoting

$$A_0 = A|_{\overline{\mathcal{R}(\operatorname{Re}T)}},$$

we infer that

$$\overline{\mathcal{R}(T^*)} = \overline{A|\operatorname{Re}T|^{1/2}\mathcal{H}} = \overline{A_0\mathcal{R}(\operatorname{Re}T)} = \overline{\mathcal{R}(A_0)} \subset \overline{\mathcal{R}(A)}.$$

In addition, if  $\mathcal{N}(T)$  reduces  $T$  then

$$\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_0)} = \overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(\operatorname{Re}T)},$$

and some relationship between the closeness of the ranges of  $T$ ,  $\operatorname{Re}T$  and  $A$  can be obtained, but unessential for the Fong-Tsui conjecture. However, from (4.2) we infer  $|T|^2 \leq A|\operatorname{Re}T|^2A^*$ . So, if  $A^*$  is a  $|\operatorname{Re}T|^2$ -contraction that is  $A|\operatorname{Re}T|^2A^* \leq |\operatorname{Re}T|^2$ , then  $|T|^2 \leq |\operatorname{Re}T|^2$  and by

[10, Theorem 1.3] it follows that  $T$  is self-adjoint. In particular, if  $A$  satisfies some conditions of self-adjointness, for instance those quoted in [13] then by (4.2)  $A$  and  $|\operatorname{Re} T|$  commute, therefore by the previous remark one has  $|T|^2 \leq |\operatorname{Re} T|^2$ . But, in general the condition (1.1) does not impose other restriction on the contraction  $A$  in (4.2). In conclusion we doubt that (1.1) for  $T$  implies always  $T = T^*$ .

Notice finally that an interesting context where it is possible to show that Fong-Tsui conjecture is true is that of  $(A, m)$ -expansive operators which were recently studied in [4] and [12], but we do not refer to them here.

Another interesting context for investigations on this conjecture is that of  $A$ -contractions (for some positive operator  $A$  on  $\mathcal{H}$ ), or even for  $A$ -bounded operators, which were extensively studied in the last years. Here is natural to use the concept of  $A$ -adjoint operator and  $A$ -projection, and an important role have the  $A$ -partial isometries recently investigated in [5, 6], [7], [9], or just the quasi-isometries in the context of  $A$ -contractions which were studied in [16].

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